Electrostatic self-energy in static black holes with spherical symmetry

B. Linet *

Laboratoire de Mathématiques et Physique Théorique

CNRS/UPRES-A 6083, Université François Rabelais

Parc de Grandmont 37200 TOURS, France

Abstract

We determine the expression of the electrostatic self-energy for a point charge in the static black holes with spherical symmetry having suitable properties.

1 Introduction

A renewed interest in the electrostatic self-energy for a point charge in black holes has appeared in studies of the upper bound on the entropy for a charged object [1, 2, 3]. For a neutral object, the original method required the validity of the generalized second law of thermodynamics for the Schwarzschild black hole [4]. In the charged case, it is moreover essential to take into account the electrostatic self-energy on the horizon of the charged object within the method. For the Schwarzschild black hole, the expression of the self-force acting on a point charge at rest has been determined for a long time [5, 6, 7] because the electrostatic potential was known in closed form [8, 9] and so the entropy bound has been found by this way. However, we have recently taken up the question of how derive the entropy bound for a charged object by employing thermodynamics of any static black holes with spherical symmetry [10]. A crucial step in this case is again the determination of electrostatic self-energy. Unfortunately, our expression of the electrostatic self-energy was mainly conjectured from some strong indications but the precise assumptions on the black holes ensuring this result was not indicated.

The purpose of this work is to determine of the expression of the electrostatic self-energy for a point charge in the static black holes with spherical symmetry verifying the following general assumptions.

1. The coordinate system in which the metric is manisfestly static must describe entirely the spacetime outside the horizon of the black hole. However, it is not necessarily asymptotically Minkowskian.

*E-mail: linet@celfi.phys.univ-tours.fr

2. The surface gravity of the horizon of the black hole is different from zero.

3. When the black hole is charged, then the electrostatic potential generated by this charge tends to zero at the spatial infinity.

4. The no hair conjecture for the electric field holds for the black hole, i.e. there is no black hole with electric multipole moments, except with the monopole.

In fact, the third and fourth assumptions induce the same properties on the test electric field in the background geometry of the given black hole. In our procedure, these assumptions have to be directly verified on the test electric multipole fields.

The plan of the work is as follows. We recall in section 2 some useful formulas about the static black holes with spherical symmetry. In section 3 we discuss the electrostatic equation in the black holes verifying the above mentioned assumptions. We calculate in section 4 the electrostatic self-energy on the horizon. We add some concluding remarks in section 5.

2 Preliminaries

If we suppose that the spacetime describing the static black hole with spherical symmetry is such that the area of the spheres increases with the radial coordinate, then its metric can be written as

$$ds^2 = g_{tt}(R)dt^2 + g_{RR}(R)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

in a coordinate system (t, R, θ, φ). The component g_{tt} vanishes at the horizon located at $R = R_H$ with $R_H > 0$. By virtue of the first assumption, metric (1) is well defined for $R_H < R < \infty$. In this domain, we have $g_{tt} < 0$ and $g_{RR} > 0$ but for a black hole g_{RR} becomes singular as $R \to R_H$. The surface gravity of the horizon κ has the general expression

$$\kappa = \left.\frac{\partial_R g_{tt}}{2\sqrt{-g_{tt}g_{RR}}}\right|_{R=R_H}. \tag{2}$$

In our procedure, we have needed to write down metric (1) in isotropic form. To do this, we perform a change of radial coordinate $R(r)$ defined by the differential equation

$$\sqrt{g_{RR}(R)}\frac{dR}{dr} = \frac{R}{r}. \tag{3}$$

It follows from (3) that the radial coordinate r is determined up to an arbitrary factor but it is well defined outside the horizon. We denote r_H the corresponding value of R_H and we remark that $r_H \geq 0$. In the coordinate system (t, r, θ, φ), metric (1) of the black hole can be now written as

$$ds^2 = -N^2(r)dt^2 + B^2(r)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right) \tag{4}$$

in which the horizon is located at $r = r_H$. In the domain $r_H < r < \infty$, metric (4) is well defined. Since the hypersurface $r = r_H$ is a horizon, we have $N(r_H) = 0$.

The second assumption, $\kappa \neq 0$, implies that $\partial_r N(r_H) \neq 0$ and $\lim_{r \to r_H} B(r)$ finite. This excludes $r_H = 0$ and $B(r_H)$ being finite, we have $B(r_H) \neq 0$ since the area of the black hole does not vanish. Hence from the general expression (2), we get

$$\kappa = \frac{N'(r_H)}{B(r_H)} \tag{5}$$

where the prime signifies the differentiation with respect to r. Taking into account (5), we see immediately that

$$N(r) \sim \kappa B(r_H)(r - r_H) \quad \text{as} \quad r \to r_H. \tag{6}$$

with $\kappa B(r_H) \neq 0$.

3 Electrostatic potential

In the static case, the Maxwell equations in background (4) yields the following equation for the electrostatic potential A_t

$$\frac{1}{\sqrt{-g}} \partial_i \left(\sqrt{-g} g^{tt} g^{ij} \partial_j A_t \right) = 4\pi J^t \tag{7}$$

where J^t is the charge density. For a point charge e at $r = r_0$, $\theta = \theta_0$ and $\varphi = \varphi_0$, it has the expression

$$J^t(r, \theta, \varphi) = \frac{e}{\sqrt{-g}} \delta(r - r_0) \delta(\theta - \theta_0) \delta(\varphi - \varphi_0). \tag{8}$$

Without loss of generality, we may write down the electrostatic equation (7) for $\theta_0 = 0$ in the form

$$\triangle A_t + \frac{N}{B} \left(\frac{B}{N} \right)' \partial_r A_t = -4\pi e \frac{N}{r^2 B} \delta(r - r_0) \delta(\cos\theta - 1) \tag{9}$$

where $\triangle$ is the usual Laplacian operator.

The electric field derived from the electrostatic potential, solution to equation (9), should be well behaved at the horizon and at the spatial infinity of the black hole. Now we require the absence of charge inside the horizon, therefore there is no electric flux through the horizon. Consequently, the Gauss theorem gives

$$\int_{r_1} \frac{\sqrt{-g}}{\sin\theta} g^{tt} g^{rr} \partial_r A_t d\theta d\varphi = \begin{cases} 4\pi e & r_1 > r_0 \\ 0 & r_H < r_1 < r_0. \end{cases} \tag{10}$$

We can expand A_t in spherical harmonics. In the axially symmetric case, we put

$$A_t(r, \theta) = \sum_{l=0}^{\infty} R_l(r, r_0) P_l(\cos\theta) \tag{11}$$

where the function R_l obeys the differential equation

$$R_l'' + \left(\frac{2}{r} + \frac{B'}{B} - \frac{N'}{N}\right) R_l' - \frac{l(l+1)}{r^2} R_l = -e(2l+1)\frac{N}{r^2 B}\delta(r - r_0). \tag{12}$$

The problem to determine R_l reduces to define two linearly independent solutions g_l and f_l of the homogeneous differential equation (12) with appropriate boundary conditions.

In the case $l = 0$ of equation (12), an integration leads to

$$\partial_r R_0(r) = \text{const.} \times \frac{N(r)}{r^2 B(r)}.$$

By virtue of our third assumption, the test electric field in background (4) generated by a charge inside the horizon has an electrostatic potential vanishing at the spatial infinity and regular at the horizon. So, we can put

$$g_0(r) = 1 \quad \text{and} \quad f_0(r) = a(r) \quad \text{with} \quad a(r) = \int_r^\infty \frac{N(r)dr}{r^2 B(r)}. \tag{13}$$

According to the Gauss theorem (10), a is the electrotatic potential generated by a unit charge inside the horizon. It is finite at $r = r_H$ and we set $a(r_H) = a_H$.

In the case $l \neq 0$ of equation (12), the point $r = r_H$ is a singularity of the homogeneous differential equation. From (6), we see that

$$\frac{N'}{N} \sim \frac{1}{r - r_H} \quad \text{as} \quad r \to r_H$$

and consequently the point $r = r_H$ is a singular point of regular type of the differential equation (12). The roots of the indicial equation relative to this point are 0 and 2. Thus, there exists a regular solution at $r = r_H$, noted g_l, such that

$$g_l(r) \sim (r - r_H)^2 \quad \text{as} \quad r \to r_H \tag{14}$$

and so the corresponding electric field is well behaved on the horizon. The solution g_l cannot regular as $r \to \infty$ because the test electric field would be well behaved at the spatial infinity and this fact would be in contradiction with the fourth assumption which demands the nonexistence of black hole with multipole electric moments, except with the monopole. Consequently, the solution g_l is singular as $r \to \infty$. We call f_l the regular solution as $r \to \infty$. By using the same argument, we find that the solution f_l is singular at $r = r_H$.

Therefore, the electrostatic potential (11) having the adequate boundary conditions can be written down in the form

$$A_t(r,\theta) = \begin{cases} e a_H + \sum_{l=1}^{\infty} e C_l g_l(r) f_l(r_0) P_l(\cos\theta) & r_H < r < r_0 \\ e a(r) + \sum_{l=1}^{\infty} e C_l g_l(r_0) f_l(r) P_l(\cos\theta) & r > r_0 \end{cases} \tag{15}$$

where the constants C_l are uniquely determined by equation (12).

We now return to the partial differential equation (9). As the second partial derivatives in this operator take the form of the usual Laplacian, the behaviour of A_t at the neighbourhood of the point $r = r_0$ and $\theta = 0$ is given by

$$A_t(r, \theta) \sim \frac{N(r_0)}{B(r_0)} \times \frac{e}{\sqrt{r^2 - 2rr_0 \cos\theta + r_0^2}}. \tag{16}$$

4 Electrostatic self-energy

We consider the electrostatic energy associated with the Killing vector ∂_t of metric (4). The Coulombian part (16) of the electrostatic potential A_t does not yield an electrostatic self-force. As shown in the previous works [5, 6, 7], the regular part of A_t at $r = r_0$ and $\theta = 0$ enables us to define the electrostatic self-energy $W_{self}(r_0)$ by the following limit process

$$\frac{e}{2}\left(A_t(r, \theta) - \frac{N(r_0)}{B(r_0)} \times \frac{e}{\sqrt{r^2 - 2rr_0 \cos\theta + r_0^2}}\right) \to W_{self}(r_0) \quad \text{as } r \to r_0 \ \theta \to 0. \tag{17}$$

However, since we do not know in general the expression of A_t in closed form, it is difficult to evaluate $W_{self}(r_0)$ by using (17).

In order to calculate (17), we consider the explicit function V_C introduced by Copson in the Schwarzschild metric characterized by the mass M [8]. It is the solution in the Hadamard sense of the electrostatic equation in isotropic coordinates, i.e. equation (9) with the coefficient

$$\frac{N^S(r)}{B^S(r)} = \frac{1 - M/2r}{(1 + M/2r)^3}.$$

We choose $M = 2r_H$ so that the horizon of the Schwarzschild black hole in isotropic coordinates coincides with r_H. This function V_C has also the same behaviour (16) in a neighbourhood of the point $r = r_0$ and $\theta = 0$ which is given by

$$V_C(r, \theta) \sim \frac{(1 - M/2r_0)}{(1 + M/2r_0)^3} \times \frac{e}{\sqrt{r^2 - 2rr_0 \cos\theta + r_0^2}}. \tag{18}$$

We are now in a position to find a new limit process, instead of (17), by replacing $e/\sqrt{r^2 - 2rr_0 \cos\theta + r_0^2}$ by $V_C(r, \theta)$ with the appropriate factor which takes into account (18). We thus have

$$\frac{e}{2}\left(A_t(r, \theta) - \frac{N(r_0)}{B(r_0)} \times \frac{(1 + M/2r_0)^3}{(1 - M/2r_0)} V_C(r, \theta)\right) \to W_{self}(r_0) \quad \text{as } r \to r_0 \ \theta \to 0. \tag{19}$$

The solutions g_l^S and f_l^S are known in function of the radial coordinate R of the Schwarzschild metric, likewise the constant C_l^S [11]. Of course, the solutions g_l^S and f_l^S

satisfy the desired boundary conditions because the no hair theorem for the Schwarzschild black hole has been proved [12]. The analysis of the explicit expression of the Copson solution V_C with the aid of the Gauss theorem (10) shows that it describes furthermore a charge $-eM/r_0(1+M/2r_0)^2$ inside the horizon [9]. In the multipole expansion (11) of V_C, the monopole term must take into account this fact. We have thereby

$$V_C(r,\theta) = \begin{cases} \dfrac{e}{r_0\left(1+M/2r_0\right)^2}\left(1-\dfrac{M}{r\left(1+M/2r\right)^2}\right) & \\ \quad +\displaystyle\sum_{l=1}^{\infty} eC_l^S g_l^S(r) f_l^S(r_0) P_l(\cos\theta) & r_H < r < r_0 \\ \dfrac{e}{r\left(1+M/2r\right)^2}\left(1-\dfrac{M}{r_0\left(1+M/2r_0\right)^2}\right) & \\ \quad +\displaystyle\sum_{l=1}^{\infty} eC_l^S g_l^S(r_0) f_l^S(r) P_l(\cos\theta) & r > r_0. \end{cases} \tag{20}$$

We now insert the multipole expansions (15) and (20) into the left term of formula (19). We simply set $r = r_0$ and $\theta = 0$ in the resulting infinite series to evaluate $W_{self}(r_0)$. Now we are only interested in the determination of the electrostatic self-energy on the horizon, denoted W_{self}, therefore we take the limit $r_0 \to r_H$ in this infinite series. The important point in the limit process is that

$$\lim_{r_0\to r_H} \frac{N(r_0)}{r_H\left(1-M/2r_0\right)B(r_H)} = \kappa \tag{21}$$

which can be derived from (6). Now each term $l = 1, 2, \ldots$ of the infinite series contains $g_l(r_H)$ or $g_l^S(r_H)$ which vanish according to (14), therefore it remains only to study in (19) the monople terms of the multipole expansions (15) and (20). Due to (21), we obtain finally

$$W_{self} = \frac{e^2}{2}(a_H - \kappa). \tag{22}$$

For another Killing vector $\partial_{\overline{t}}$ resulting from the rescaling of the time coordinate $\overline{t} = \lambda^2 t$, we easily see that $\overline{W}_{self}$ is given W_{self}/λ since $\overline{a}_H = a_H/\lambda$ and $\overline{\kappa} = \kappa/\lambda$. If metric (4) is asymptotically Minkowskian, then we can normalize ∂_t.

5 Conclusion

For black holes verifying the prescribed assumptions, we have provided a method for the calculation of the electrostatic self-energy on the horizon, fortunately without having to know the expression of the electrostatic potential in closed form. The key point is the limit process (19). We emphasize that only the knowledge of the monopole in the multipole expansion of the electrostatic potential is required to calculate the electrostatic self-energy.

From (22), we can immediately show that the electrostatic self-energy at the position $r = r_0$ is given by

$$W_{self}(r_0) = \frac{1}{2}e^2 s[a(r_0)]^2 \quad \text{with} \quad s = \frac{1}{a_H}\left(1 - \frac{\kappa}{a_H}\right) \tag{23}$$

which agrees with our previous conjecture [10]. Of course, this formula is independent on the choice of the radial coordinate.

In the present proof, we have required that the surface gravity of the horizon κ is different from zero. It is probably not necessary for all extreme black holes. Indeed, formula (22) or (23) gives the electrostatic self-energy in the extreme Reissner-Nordström black hole for which $\kappa = 0$. We can directly see it from the expression of the electrostatic potential in the Reissner-Nordström black hole which is known in closed form [13].

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